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Chapter-2

M.Sc.

Type of operators

Following are the important type of operators.

1. Linear operators. An operator is said to is to be liner if it operated on the sum of two or more functions and give the result as sum of the results of operation on each function.

e.g. if A is applied on the sum of two functions $\Psi(x)$and $\Phi(x)$, linear operator gives

Ex: $\frac{\partial}{\partial}(x^3)$ =$3x^3$

Ex: $\hat{A}[\frac{\partial^2}{\partial\ ^2}+2\frac{\partial}{\partial}+3]$ the $\hat{A}(x^x)=\frac{\partial^2}{\partial\ ^2}(x^3)+2\frac{\partial}{\partial}x^3+3x^3$

= 6x+$6x^2$+$3x^3$

$$A\,[\Psi(x)+\Phi(x) \neq \sqrt{\Psi(x)}+\sqrt{\Phi(x)}$$

Similarly operator which squares the function or raise the power by three or any number is not liner. But differential operator is liner.

i.e. $[\Psi(x)+\Phi(x)]$=

2. Eigen operators. An operator say A is said to be an Eigen operator if it operator if it operator on a function say $\Psi(x)$giving a which is the original function multiplied by some constant.

Following quality should exit.

$$A\ \Psi(x) = \alpha\Psi(x)$$

Ex: H Ψ =E Ψ

Ex : e^{ax} is a eigen function of $\frac{\partial}{\partial}, \frac{\partial^2}{\partial\ ^2}, \frac{\partial^n}{\partial\ ^n}$

$\frac{\partial^n}{\partial\ ^n}(e^{ax})=a^n e^{ax}$ a^n is eigen valve ,

Addition of operator& Subtraction :- $(\hat{A}+\hat{B})$ f(x) ^ =Af(x)=±Bf(x)

$(\hat{A}+\hat{B})$f= $\hat{A}$(F)+B(F)

Ex: $[\frac{\partial^2}{\partial\ ^2}+\frac{\partial^2}{\partial\ ^2}]$ F(x,y)= $\frac{\partial^2}{\partial\ ^2}$F(x,y) = $\frac{\partial^2}{\partial\ ^2}$ F(x,y)+ $\frac{\partial^2}{\partial\ ^2}$f(x,y)

<u>Subtraction of operator</u>

$(\hat{A}-\hat{B})$f(x) = $\hat{A}$f(x)-$\hat{B}$f(x)

Ex: $\hat{A}$ be loge &$\hat{B}$ be $\frac{d}{d}$ and f(x) be x^2 then .

<u>Multiplication of operator</u> :

$\hat{A}\hat{B}$ f(x) = A $fI(x)$ = f^{II}(x)

Using operator from righty to left .

Ex : A^2 f(x) ≠ $[\hat{A}$ f(x)$]^2$

$\frac{\partial^2}{\partial\ ^2}(5x^2+3x)\ \frac{\partial}{\partial}\left\{\frac{\partial}{\partial}(5x^2+3x\right\}=\frac{\partial}{\partial}(10x+3)=10.$

$[\hat{A}\, l(x)]^2 \left\{\frac{\partial}{\partial}(5x^2+3x)\right\}^2$ =100x²+9+60X

The constant 'α' is called Eigen value and function $\Psi(x)$ is called Eigen function or *characteristic function or proper function of the operator* A.

For example, when an operator $\frac{d^2}{dx^2}$ Is carried out on function *cos 2 x.*

$-\frac{d^2}{dx^2}(\cos 2x) = 4(\cos 2x)$

In this case cos2 x is the eigen function and 4 is **Eigen value** of the operator $-\frac{d^2}{dx^2}$.

3. **Hermitian operator.** If $\hat{A}$ an Eigen operator is on the functions $\Psi_1(x)$ and $\Psi_2\ (x)$ then it is said to be Hermitian if the following equation holds valid.
   $\int \Psi 2^{\star} \hat{A}(\Psi) d = \int \Psi \hat{A}^{\star}(\Psi 2^{\star})\, d$

Hermintian operator always yields real eigen valve. As shown below. If Ψ1 = Ψ2 = Ψ the condition for Hermintian nature is $\int \Psi^* \hat{A}(\Psi) d\tau$

Let 'a' be the eigen valve of the operator $\hat{A}$ for the eigen function Ψ

∴ $\hat{A}\Psi = a\Psi$

Multiplying both sides by Ψ ∗ and integrate

$$\int \Psi^* \hat{A}(\Psi) d\tau = \int \Psi A^{\star}(\Psi 2^{\star})\, d = A \int \Psi * \Psi\, d\tau$$

Taking complex conjugate of each in eigen valve equation

$\hat{A} * \Psi^* = a^* \Psi^*$

Multiplying both sides by Ψ and integrating

$$\int \Psi^* A(\Psi *) = \int \Psi a^* \Psi * d\tau = a^* \int \Psi\Psi * d\tau$$

Because of Hermintian nature L.H.S of (A) and (B) ar equal. Therefore. R.H.S of (A) and (B) are also equal.

∴ $a \int \Psi\Psi * d\tau = a^* \int \Psi\Psi * d\tau$ or a = a⋆

Which is true when th3e constant a, the eigen valve of the Hermitian operator is a real number because only real number equal their complex conjugate. It may be noted that for allowed eigen function. The eigne valve are always real and the corresponding liner eigen Hermintian operator represets and observable .

Note. All quantum mechanical operator must be linear and herminatian in the nature .

4. **Unitary operator**. An eigen operator is said to be unitary operator. $\hat{a}$ if it operates on eigen function Ψ1 and Ψ2 such that

$$\int \Psi \star (\hat{a}^{-1} \Psi)\, d\tau = \int \Psi 1 (\hat{a}^{-1} \Psi \star)\, d\tau$$

Where $\hat{a}^{-1}$ is the inverse of unitary operator $\hat{a}$ such that $\hat{a}\hat{a}^{-1} = a^{-1} = 1$.

The asteric stands for the complex conjugate functions.

When the two wave function Ψ1 and Ψ2 are same i.e., Ψ1 = Ψ2 = Ψ the above equation reduces to

$\int \Psi \star(\hat{a}^{-1} \Psi)\, d\tau = \Psi\, (a^{\star}\, \Psi \star)\, d\tau$
...(i)

Since a is eigen operator, let a Ψ = A Ψ

Where A is the eigen valve

Operating the operator a on eqa ation (i)

$\hat{a}^{-1} a\, \Psi = \hat{a}^{-1}\, \Psi - A\hat{a}^{-1}\, \Psi$

∴ $\Psi = A a^{-1}\, \Psi$

Or $\hat{a}^{-1}\Psi = A^{-1}\, \Psi$
.....(ii)

This implies that the eigen valve of the inverse operator $\hat{a}^{-1}$ is the reciprocal (A^{-1}) of the eigen valve (A) of simple operator $\hat{a}$.

Multiplying both sides of the equation (ii) by Ψ ⋆ and integrating ove th3e whole space.

$\int \Psi \star a^{-1}\, \Psi d\tau \;=\; \int \Psi \star A^{-1}\, \Psi\, d\tau \;=\; = A^{-1}\, \Psi \star\, \Psi\, d\tau$
...(iii)

Taking the3 complex conjugate of each quantity in eqn. (i)

Multiplying eqn. (iv) by Ψ on both sides and integrating over all the space.

$\int \Psi \star \hat{a}^{*}\, \Psi * d\tau = \int \Psi A^{*}\, \Psi\, d\tau = = A^{*}\, \Psi\, \Psi^{*} d\tau$
....(v)

According to the condition for the unitary operator that left hand side of equation (iii) and (v)are equal

Hence $A^{-1} \int \Psi\, \Psi^{*}\, d\tau = A^{*}\, \Psi\, \Psi^{*}\, d\tau$ or $A^{-1} = A^{*} A = 1$

Or | A^2 | 1 or A=1

5. **Laplacian Operator**. It represented by $\nabla^2 = \frac{\partial^2}{\partial\ ^2} + \frac{\partial^2}{\partial\ ^2} + \frac{\partial^2}{\partial\ ^2}$
   This is common operator in quantum mechaniecs.
   The schrodinger equation.
   $\nabla^2 = \frac{\partial^2}{\partial\ ^2} + \frac{\partial^2}{\partial\ ^2} + \frac{\partial^2}{\partial\ ^2}$
   It is is used in quantum mechanic.
   There schrodinger equation viz
   $\frac{\partial^2\Psi}{\partial\ ^2} + \frac{\partial^2\Psi}{\partial\ ^2} + \frac{\partial^2\Psi}{\partial\ ^2} + \frac{8r^2 m(E-V)}{h^2}\Psi = 0$
   It terms of Laplacian operator may be written as

$\nabla^2\, \Psi + \frac{8r^2 m(E-V)}{h^2}\, \Psi = 0$

6Hamiltonian operator. The schrodinger equation∇^2
$\Psi \frac{8r^2 m(E-V)}{h^2}\, \Psi = 0$

May be written as $\nabla^2\, \Psi = -\frac{8r^2 m(E-V)}{h^2}\Psi = -\frac{8r^2 m}{h^2}(E\Psi - V\Psi)$

Or $-\frac{h}{8r^2 m}^2 \nabla^2\Psi + V\Psi = E\Psi$

The above equation indicates that the operation $(\frac{h^2}{8r^2 m}\nabla^2 + V)$ is carried on the function Ψ.

The operator $\frac{-h}{8r^2 m}^2 \nabla^2 + V$ is called Hamiltonian operator and is represented by $\hat{H}$

∴ $\hat{H} = (\frac{-h}{8r^2 m}^2\, \nabla^2)$

The3 schrodinge wave equation maybe written in another short form in terms of Hamiltonian operator.

$$\hat{H}\,\Psi = E\,\Psi$$

Her Ψ is the eigen function an E is the eigen valve.

Equal operator :- The two operators say $\hat{A}$ and $\hat{B}$ are ca;;ed equal operator, If they operating on a function separately produces the same result

Ex : $\frac{\partial^2}{\partial_x \partial_y} x^2 y = 2x \quad \frac{\partial^2}{\partial_y \partial_x} x^2 y = 2x$

Hence $\frac{\partial^2}{\partial_x \partial_y}$ and $\frac{\partial^2}{\partial_y \partial_x}$ are equal operator

Commutation of operator :- operators $\hat{A}$ and $\hat{B}$ commute when, $\hat{A}\hat{B}f(x)=\hat{B}\hat{A}f(x)\ \left[\hat{A}\hat{B}-\hat{B}\hat{A}\right] f(x) = 0$

$\left[\hat{A}\hat{B}-\hat{B}\hat{A}\right]$ is represented by $\left[\hat{A}\hat{B}\right]$ and is called commutation of the operators $\hat{A}$ and $\hat{B}$. If $\hat{A}$ and $\hat{B}$ commute, then $\left[\hat{A},\hat{B}\right]=0$ otherwise $\left[\hat{A},\hat{B}\right] \neq 0$

Ex: $\hat{A} = 5$, $\hat{B} = \frac{\partial^2}{\partial_x{}^2}$ of $f(x) = x^2 + 2x$

$\hat{A}\hat{B}$ (x2+2x)=5.$\frac{\partial^2}{\partial_x{}^2}(x^2+2x)=5.2=10$

$\hat{B}\hat{A}\ (x^2+2x) = \frac{\partial^2}{\partial_x{}^2} 5(x^2+2x) = \frac{\partial^2}{\partial_x{}^2}(5x^2+10) = 10.$

Hence $\left[\hat{A}\hat{B}-\hat{B}\hat{A}\right]=0$ and $\hat{A},\hat{B}$ are said to be commute .

<u>Postulates of quantum mechanies</u>

(1) For even time independent state of system, a function Ψ of the co- ordinates can be written which is single valued , continuous and finite through out the configuration space . The function describe completely the state of the system . This function called wave function or state function, has the properly that $\Psi(x_1,x_2\ldots t)\ \Psi(x_1,x_2\ldots.t)dz$ represents the probability of finding the system in the small volume dt of the space at time t $\int \Psi(x_1,x_2\ldots\ldots)\ \Psi(x_1,x_2\ldots.)dz = 1$
Space.

(2) T0 each observable quantity in classical mechanics. Like position, velocity, momentum enersly etc. There corresponds a certain mathematical operator in quantum mechanics.

Position	x	$\hat{x}$
Momentum	Px	$\hat{P}_x \quad \frac{h}{2\pi}, \frac{\partial}{\partial}$
Momentum square	$\hat{P}_x$	$-\frac{h}{4\pi^2}, \frac{\partial^2}{\partial\ ^2}$
Kinetic enersy	$T = \frac{p\ ^2}{2m}$	$\hat{T}_x\ -\frac{h}{8\pi^2 m}, \frac{\partial^2}{\partial\ ^2}$
Potential enersy	V(x)	$\hat{V}(x)$ multiplication by V(x)
Total enersy	$E - T_x + V(x)$	$\hat{H}\ -\frac{h^2}{8\pi^2 m}, \frac{\partial^2}{\partial\ ^2}$ +V(x)

Postulate (3) :- in any measurement of the observable properly corresponding to the operator $\hat{A}$, the only vlue that will ener be measured are the eiglen value a; which satisfy the eiglen value equality,

$$\hat{A}\Psi(x,t) = a,\Psi i(x,t)$$

a, ⟶ eigln value d Ψi (x, t) is eigln function

Postulate (4) when a large No. of identical systems have the same wave function Ψ and with each system the observable property A is measured once, then the average value of property for all of these measurements is given as

<A> =∫Ψ^* $\hat{A}$ Ψdx/∫Ψ^* Ψ dx
all space all space

Postulate (5) – when sate of the system be comes time dependent, I,e Ψ and the hamiltoniar it are time dependent, then the wave function satisfy the time dependent schrodinger equation.

$$\hat{H}\Psi(x,t) = \frac{\partial\Psi(x,t)}{\partial}$$